

Distribution of Matrix Elements of Random Operators*

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Abstract

It is shown that an operator can be defined in the abstract space of a random matrix ensemble whose matrix elements statistical distribution simulates the behavior of the distribution found in real physical systems. It is found that the key quantity that determine these distributions is commutator of the operator with the Hamiltonian. Application to symmetry breaking in quantum many-body system is discussed.

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Ensembles of random matrix theory (RMT) [1–3] have had a wide application as models to describe statistical properties of eigenvalues and eigenfunctions of chaotic many-body systems. More general ensembles have also been considered [4], in order to cover situations that depart from the conditions of applicability of RMT. One such class of ensembles is the so-called deformed Gaussian orthogonal ensemble (DGOE) [5]. These "intermediate" ensembles are particularly useful when one wants to study the breaking of a given symmetry in a many-body system such as the atomic nucleus. Further, different variances of the deformed random matrix ensembles can interpolate any one of the three universal RMT ensembles, i.e., the GOE, the GUE and the GSE, and the Poisson ensemble, which would represent the transition between a fully chaotic situation, with no conserved quantity but energy, to a regular one, i.e., one with a complete set of operators that commute with the Hamiltonian. When discrete symmetries such as isospin, parity and time reversal are violated in a complex many-body environment, one relies on a description based on the transition between one GOE [6,7] into two coupled GOE's in the first two cases and a GOE into GUE in the last case. This latter case has recently been studied by us in the case of disordered metal-insulator transition [8].

The above mentioned statistical properties refer to fluctuations, around average values, of quantities connected to the eigenvalues and the eigenfunctions. These mean values are specific to the physical system being considered and, in fact, semiclassical estimates of them can be derived in terms of the underlying Hamiltonian [9]. The fluctuations, on the other hand, have a quantum origin and are, in principle, universal, in the sense that the only information they carry about the system, is the class of underlying symmetry they belong. In order to compare the statistics generated by these fluctuations with the predictions of the ensembles, it is therefore necessary, in both cases, of eigenvalues and eigenfunctions, to perform some convenient rescaling of variables that eliminates their average behavior. In the case of the eigenvalues, they are first unfolded, which means that they are mapped onto new levels with a constant density made equal to one. For the eigenfunctions, the most convenient quantities to be statistically analyzed are not directly the components of the eigenfunctions, taken with respect to some particular basis, but rather matrix elements of a given operator. These matrix elements also show statistical fluctuations around their mean values [9] and they have to be subjected to some local average process that extracts secular variations as a function of the energy [10]. However, what is still lacking is a direct comparison of matrix elements distributions with ensemble calculations. The difficulty being that, except in the limiting case of fully chaotic regime, nothing has been so far done in random matrix ensembles studies about these distributions. In the chaotic situation, with no conserved quantities and, as consequence, without any active selection rule, the matrix elements behave like components of an isotropic random vector and one expects them to follow the same distribution of the eigenstate components, i.e., the Porter-Thomas law [11]. Nevertheless, this argument does not hold in intermediate situations between chaos and order.

Of course, the reason why matrix elements distributions have not yet been investigated, in the context of matrix ensembles, is simple: it is not clear how one can define an operator associated to an observable in this abstract space. This is exactly the question that is being addressed in this letter. We want to show that it is possible to introduce some random operator whose matrix elements would simulate, in some way, the behavior of observables found

in calculations and measurements performed in real physical systems. In the construction of this operator, we will be guided by the idea that when a system undergoes a chaos-order transition, a quantity that has a key role in determining the statistical behavior of the matrix elements of an operator, is the value of its commutator with the Hamiltonian. This is implied by the equation

$$(E_l - E_k) \langle E_l | O | E_k \rangle = \langle E_k | [O, H] | E_k \rangle = i\hbar \frac{d}{dt} \langle E_l | O | E_k \rangle \quad (1)$$

here the last equality was obtained using Schrödinger equation and assuming an operator with no explicit dependence on time. Eq.(1) clearly shows that the commutator supplies the connection between the matrix elements of an observable and its behavior as a function of time.

To see how this is reflected in the statistical distribution, suppose that we choose to look at the matrix elements of an observable O which becomes a conserved quantity in the regular regime. The distribution of these elements will undergo a transition from the Porter-Thomas law, at the chaotic side, to the singular distribution $\langle E_l | O | E_k \rangle \propto \delta_{kl}$, at the regular side, since the last term in (1) is zero in this case. In a more general situation, as it occurs when reduced transition strengths are measured, what happens is that some transitions might become forbidden at the less chaotic regime. Then as the selection rules becomes operative, the first term in (1) vanishes for some pairs (l, k) and, as a consequence, we may say that we have a partial conservation of the observable which is causing the transitions. Again one should expect the missing transitions to cause a deviation of the statistical distributions from the Porter-Thomas law.

To define the ensembles of random matrices we are going to work with, we follow the construction based on the Maximum Entropy Principle [5], that leads to a random Hamiltonian which can be cast in the form

$$H = H_0 + \lambda H_1, \quad (2)$$

where λ is the parameter that controls the chaoticity of the ensemble. We will assume it to be defined in the domain $0 \leq \lambda \leq 1$, in such a way that for $\lambda = 1$, $H = H^{GOE}$, and for $\lambda = 0$, we have some reduced chaotic situation defined by the choice of H_0 . Since we are specifically interested in the transitions from GOE to Poisson and from GOE to two coupled GOE's, the above requirements are sufficient to determine H_0 and H_1 .

First, we consider the GOE→Poisson transition, in which case we write [12]

$$H_0 = \sum_{i=0}^N P_i H^{GOE} P_i \quad (3)$$

and

$$H_1 = \sum_{i \neq j}^N P_i H^{GOE} P_j \quad (4)$$

where H^{GOE} is a N - dimensional random matrix taken from the Gaussian Orthogonal Ensemble, and we have introduced the projection operators $P_i = |i\rangle\langle i|$, $i = 1, \dots, N$.

It is straightforward to verify, from the usual properties of projectors, that $H = H^{GOE}$ for $\lambda = 1$ and, on the other hand, when $\lambda = 0$, H becomes a diagonal matrix whose eigenvalues are known to follow Poisson distribution [1].

Considering now the GOE \rightarrow 2GOE's transition, we write [5,6]

$$H_0 = PH^{GOE}P + QH^{GOE}Q \quad (5)$$

and

$$H_1 = PH^{GOE}Q + QH^{GOE}P \quad (6)$$

where $P = \sum_{i=1}^M P_i$ and $Q = 1 - P$. Here H_0 is a two blocks diagonal matrix of dimensions M and $N - M$ and each block is by construction a GOE random matrix. Again, it is easily verified that $H = H^{GOE}$ for $\lambda = 1$.

Turning now to the operator, we choose it to have the form

$$O = \sum_{i=0}^N P_i H^{GOE} P_i. \quad (7)$$

In the case of the transition towards Poisson, since $O = H_0$, we immediately derive the commutator relation

$$[H, O] = \lambda[H_1, O].$$

where the commutator on the RHS has matrix elements given by

$$[H_0, H_1]_{ij} = (H_{ii}^{GOE} - H_{jj}^{GOE}) H_{ij}^{GOE} \quad (8)$$

which obviously is a nonvanishing antisymmetric random matrix. Therefore, the statistical behavior of the elements of O is controlled by λ and it evolves from the Porter-Thomas distribution to a singular delta distribution as λ goes from 1 to 0.

On the other hand, for the transition to two GOE's, it is convenient to separate the sum in Eq. (7) into two parts in which the first M terms define the operator O_P and the other $N - M$ define the operator O_Q , by construction $O = O_P + O_Q$. The commutator with the Hamiltonian can then be written as

$$[O_P, PHP] + [O_Q, QHQ] + \lambda(O_P PHQ + O_Q QHP - QHPO_P - PHQO_Q) \quad (9)$$

These terms have a simple interpretation. The first and the second ones are responsible for the transitions between states inside the blocks PHP and QHQ , respectively. On the other hand, the four terms inside the parenthesis cause transitions among states located inside different blocks. Since the latter terms in (9) are all multiplied by the parameter λ , when $\lambda \rightarrow 0$ these transitions become forbidden. This property shows that the operator we have introduced is very convenient to study transition towards two coupled GOE's, a scenario appropriate to investigate discrete symmetry violation in complex quantum systems.

To make our model more flexible we are going to consider, in this case of the transition to two GOE's, matrix elements of the generic operator

$$O' = (1 - q) H_0 + qO$$

where H_0 is given by Eq. (5) and q varies between 0 and 1. With this form O' represents an operator which has a conserved part. We have now a model with two parameters, the parameter λ of the Hamiltonian which may be fixed by the fitting the eigenvalues distribution and the parameter q that selects the operator. For $q = 1$, we have just a selection rule, as q decreases we are introducing a localization in the matrix elements inside the selection rule.

Following the standard procedure, we first construct, with the operator O , the normalized vector

$$| \alpha_k \rangle = \frac{O | E_k \rangle}{\langle E_k | O^2 | E_k \rangle}$$

where $| E_k \rangle$ with $k = 1, \dots, N$ is an eigenvector of the Hamiltonian. From these N vectors we define the matrix elements

$$T_{kl} = \langle E_l | \alpha_k \rangle$$

which are the quantities to be statistically analyzed.. It is convenient to work with $| T_{kl} |^2$, and, as mentioned above, perform a local average that extracts secular variation with the energies. Thus we introduce the quantities

$$y_{kl} = \frac{| T_{kl} |^2}{\langle | T_{kl} |^2 \rangle}$$

where the average is done by using a Gaussian filter of variance equal to 2. It has become standard in the analysis of these quantities to histogram their logarithm. In the Figs. 2 and 3, it is shown the numerical results obtained for the two transitions.

In Fig. 1, calculations performed in the case of the GOE-Poisson transition are presented. The histograms of the logarithm of the matrix elements for four different values of the chaoticity parameter λ are plotted together with three theoretical distributions: the Porter-Thomas, corresponding to a $\chi^2(\nu)$ distribution with $\nu = 1$ degree of freedom, a $\chi^2(\nu)$ where the degree of freedom ν is derived directly from the "data" [10] and finally a distribution in which the histograms were fitted with the superposition of two χ^2 distributions. Whereas one χ^2 distribution of the type suggested in Ref. [10], is constrained to a peak always around zero, our calculations, however, suggest the need of a linear combination of two distribution as we have already proposed in Ref. [6]. This kind of behavior has seems to be typical of transitions in which the eigenstates become more and more localized. It can be understood as a signature of the multifractal nature of the states [13].

In Fig. 2, the results for the transition towards two GOE's are shown. The chaoticity parameter was fixed at value $\lambda = 0.032$ and the parameter q , that measures the localization inside the selection rule, is varied. Again, the same theoretical distributions of Fig. 1 are also shown. At this value λ , we expect to be near the case in which the two GOE's are completely decoupled. We see from the figure that the distributions are greatly dependent on the parameter q .

We observe that in the extreme situation when we have two diagonal uncoupled blocks, the probability distribution can be written down explicitly. In fact, since inside each block

we have Porter-Thomas distributions and outside of them the matrix elements vanish, we have for a generic reduced strength $y = |T_{kl}|^2$, the distribution

$$P(y) = \left(\frac{M_1}{N}\right)^2 \sqrt{\frac{\pi M_1}{2y}} \exp\left(-\frac{M_1 y}{2}\right) + \left(\frac{M_2}{N}\right)^2 \sqrt{\frac{\pi M_2}{2y}} \exp\left(-\frac{M_2 y}{2}\right) + 4\frac{M_1 M_2}{N^2} \delta(y)$$

where $M_1 + M_2 = N$. The case $q = 0$, in Fig.2, corresponds, for the above value of the chaoticity parameter, to an almost uncoupled situation. We see that we have only a slight deviation from Porter-Thomas. This means that the delta function, in the above expression, is practically not observed. This can be explained by the fact that we are plotting the logarithm of the intensities. In this kind of plot the zero elements are not observed and, more than that, they gain a vanishing statistical weight that comes from the Jacobian of the transformation $y \rightarrow \ln y$. Physically this means that the selection rule is hardly detected in this kind of analysis if what is being considered are statistics of matrix elements of an operator without a conserving part.

In conclusion, we have extended in this paper the Maximum entropy theory to the case of the distribution of matrix elements. Contrary to what has been suggested recently in the literature, we find that in the intermediate situation described by the deformed Gaussian ensembles, the distribution is like a sum of two χ^2 ones. Our theory is quite fit to address the question of symmetry breaking in complex many-body systems. The application to isospin symmetry violation in light nuclei [14] is underway [15].

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Figure Captions:

Fig. 1 Four histograms of the logarithm of the matrix elements distributions of the random operator O , see text, in the case of the transition $\text{GOE} \rightarrow \text{Poisson}$, for the indicated values of the chaoticity parameter λ . The calculations were done with matrices of dimension $N = 100$.

Fig. 2 Four histograms of the logarithm of the matrix elements distributions of the random operator O' , see text, in the case of the transition $\text{GOE} \rightarrow 2\text{GOE}$'s, for the indicated values of the parameter q . The calculations were done with matrices of dimension $N = 120$ and block sizes $M_1 = 40$ and $M_2 = 80$.